University of Saskatchewan
Department of Mathematics and Statistics
Math 224 (02, G. Patrick)

Friday March 24, 2006 | Test #2 | 60 minutes

This examination consists of two parts. Part A contains short, routine questions, which you should answer fully but succinctly in the space provided. The questions in Part B are more difficult, and some are designed to challenge you. Fully answer all questions of Part B in the space provided.

If you require more than the alloted space then use the back of the question sheet.

You should complete Part A rapidly, and save about half your time to answer the questions in Part B. Part A is worth 18 points and Part B is worth 12 points. Remember to print your name and student ID in the spaces provided in both Part A and Part B.

The points for each problem are indicated in the right margin.

Permitted resources: none. No books, no notes of any kind, no calculators, no electronic devices of any kind.

This is a midterm test. Cheating on a test is considered a serious offense by the University and can be met with disciplinary action, including suspension or expulsion. Candidates shall not bring into the test room any books, resources or papers except at the discretion of the examiner or as indicated on the examination paper. Candidates shall hold no communication of any kind with other candidates within the examination room.

PRINT your NAME and STUDENT ID: ____________________

PART A. Fully answer the following questions in the space provided.

Question A1. Find the open interval of convergence of the series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}(x+1)^n$. 3

$R = \lim_{n\to\infty} \frac{n^2}{3^n} \cdot \frac{3^{n+1}}{(n+1)^2} = 3$ so the open interval of convergence

is $(-1-3, -1+3) = (-4, 2)$

Question A2. Find the Taylor series up to and including order three (that is up to and including terms like $(x-\pi/4)^3$) of $f(x) = x\sin x$ centered at $x = \pi/4$. 3

$f(x) = x\sin x \quad f(\frac{\pi}{4}) = \frac{\pi}{4}\frac{1}{\sqrt2}$

$f'(x) = \sin x + x\cos x \quad f'(\frac{\pi}{4}) = \frac{1}{\sqrt2} + \frac{\pi}{4}\frac{1}{\sqrt2} = \frac{1}{\sqrt2}\left(1+\frac{\pi}{4}\right)$

$f''(x) = 2\cos x - x\sin x \quad f''(\frac{\pi}{4}) = 2\frac{1}{\sqrt2} - \frac{\pi}{4}\frac{1}{\sqrt2} = \frac{1}{\sqrt2}\left(2-\frac{\pi}{4}\right)$

$f'''(x) = -3\sin x - x\cos x \quad f'''(\frac{\pi}{4}) = -3\frac{1}{\sqrt2} - \frac{\pi}{4}\frac{1}{\sqrt2} = \frac{-1}{\sqrt2}\left(3+\frac{\pi}{4}\right)$

$$f(x) = \frac{\pi}{4\sqrt2} + \frac{1}{\sqrt2}\left(1+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + \frac{1}{2\sqrt2}\left(2-\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt2}\left(3+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^3 + \dots$$

Question A3. Find the Taylor series centered at $x = 0$ of $f(x) = (1+x)^{-1/3} + (1-x)^{-1/3}$. 3

$$(1+x)^{-\frac{1}{3}} = 1 + \frac{\left(-\frac{1}{3}\right)x}{1!} + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!}x^3 + \cdots$$

$$(1-x)^{-\frac{1}{3}} = 1 - \frac{\left(-\frac{1}{3}\right)}{1!}x + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}x^2 - \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!}x^3 + \cdots$$

$$(1+x)^{-\frac{1}{3}} + (1-x)^{-\frac{1}{3}} = 2 + 2\,\frac{(1)(4)}{3^2\cdot 2!}x^2 + 2\,\frac{(1)(4)(7)(10)}{3^4\cdot 4!}x^4 + \cdots + \frac{2(1\cdot 4\cdot 7\cdot 10\cdots(6m-5)(6m-2))}{3^{2m}\,(2m)!}x^{2m} + \cdots$$

Question A4. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n}{4^n} x^{2n+1}$. 3

Put $u = x^2$ so the series is $x\sum_{n=1}^{\infty} \frac{n}{4^n} u^n$. This radius of convergence of the u series is $\lim_{n\to\infty} \frac{n}{4^n}\cdot\frac{4^{n+1}}{n+1} = 4$, so this converges for $-4 < u < 4$

$\Leftrightarrow \ |x^2| < 4 \ \Leftrightarrow \ |x| < 2$. So $R = 2$.

Question A5. Beginning with the Taylor series of e^t centered at $t = 0$, compute a series for the definite integral $\int_0^1 e^{-x^2}\,dx$. 3

$$\int_0^1 e^{-x^2}dx = \int_0^1 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

$$= 1 - \frac{1}{3\cdot 1!} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,n!}$$

Question A6. Compute the Taylor series of $f(x) = (1-x^2)^{1/3}(1+x)^{3/2}$ centered at $x = 0$, up to and including order 3. 3

$$(1-x^2)^{\frac{1}{3}} = 1 - \frac{1}{3}x^2 + \cdots$$

$$(1+x)^{3/2} = 1 + \frac{3}{2}x + \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)x^2 + \frac{1}{6}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)x^3 + \cdots$$

$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \cdots$$

$$(1-x^2)^{\frac{1}{3}}(1+x)^{3/2} = 1 + \frac{3}{2}x + \left(-\frac{1}{3} + \frac{3}{8}\right)x^2 + \left(-\frac{1}{2} - \frac{1}{16}\right)x^3 + \cdots$$

$$= 1 + \frac{3}{2}x + \frac{1}{24}x^2 - \frac{9}{16}x^3 + \cdots$$

Math 224 Test #2 PART B. (02, G Patrick)

PART B. Fully answer the following questions in the space provided.

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Question B1. Divide the Taylor series for $\sin x$ at $x = 0$ by the Taylor series for $\cos x$ at $x = 0$, and using this, calculate the limit 4

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}.$$

Do not use L'Hospital's rule.

$$\begin{array}{r|l} & x + x^3/3 + \cdots \\ \hline 1 - \frac{x^2}{2} + \cdots & x - \frac{x^3}{6} + \cdots \\ & \underline{x - \frac{x^3}{2}} \\ & \quad \frac{1}{3}x^3 + \cdots \end{array}$$

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{x + x^3/3 + \cdots - x}{x^3} = \lim_{x \to 0} \frac{1}{3} + \cdots = \frac{1}{3}$$

Question B2. Write down the Taylor expansion of e^x about $x = 0$. How many terms of this series are required to compute, for any $x \in [-2, 0]$, the value of e^x to an accuracy of .05? 4

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$$|R_n| = \left| \frac{e^{t}}{(n+1)!} x^{n+1} \right| \le \frac{1 \cdot 2^{n+1}}{(n+1)!} = \frac{2^{n+1}}{(n+1)!}$$

n	$2^{n+1}/(n+1)!$
1	$2^2/2! = 2$
2	$2 \times 2/3 = 4/3$
3	$\frac{4}{3} \times \frac{2}{4} = \frac{2}{3}$
4	$\frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$
5	$\frac{4}{15} \times \frac{2}{6} = \frac{4}{45} > \frac{1}{20}$
6	$\frac{4}{45} \times \frac{2}{7} = \frac{8}{305} < \frac{1}{20}$

So 7 terms are required, because the truncation is at order 6 and there is a constant term.

Question B3. **Give examples of power series which satisfy each of the separate conditions (a), (b), (c), (d).** Note: you need not use the same power series for different conditions! These are four different sub-questions.

(a) A power series that converges at $x = 1$ and at $x = 2$ and has radius of convergence 1.

(b) A power series which has radius of convergence $\pi = 3.14159\cdots$.

(c) Two power series $\sum_{n=1}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} b_n x^n$, each with radius of convergence 1, such that the power series $\sum_{n=1}^{\infty} (a_n + b_n) x^n$ has an infinite radius of convergence.

(d) A power series whose radius of convergence is zero.

(a) $\sum_{n=0}^{\infty} (x - 3/2)^n$

(b) $\sum_{n=0}^{\infty} \frac{x^n}{\pi^n}$

(c) $a_n = 1,\ b_n = -1$ so $\sum_{n=1}^{\infty} (a_n + b_n) x^n = 0 + 0 \cdot x + 0 \cdot x^2 + \cdots$

which has $R = \infty$

(d) $\sum_{n=0}^{\infty} n!\, x^n$